Example Does $\sum_{n=1}^{\infty} ne^{-n}$ converge?

If $f(x) = xe^{-x} \geq 0$ for $x > 1$.

$$f'(x) = e^{-x} - xe^{-x}$$
$$= (1-x)e^{-x} < 0 \text{ for } x > 1.$$

So $f(x)$ is decreasing for $x > 1$.

We can test convergence of $\sum_{n=1}^{\infty} f(n)$ by testing convergence of $\int_1^{\infty} xe^{-x}\,dx$.

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\left(-xe^{-x}\Big|_1^b\right) + \int_1^b e^{-x}\,dx$$

parts
$u = x \quad du = dx$
$dv = e^{-x}dx \quad v = -e^{-x}$

$$= \lim_{b\to\infty} -be^{-b} + e^{-1} + -e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} -be^{-b} + e^{-1} - e^{-b} + e^{-1} = 2e^{-1} < \infty.$$

$\frac{b}{e^b} \to 0 \qquad \frac{1}{e^b} \to 0$ converges.

By the integral test, $\sum_{n=1}^{\infty} ne^{-n}$ converges.

$$\geq \sum_{n=1}^{\infty} (f(n)) \geq \int_1^{\infty} f(x)\,dx$$

$$\sum_{n=2}^{\infty} f(n) \le \int_1^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{\infty} f(n) \le f(1) + \int_1^{\infty} f(x)\,dx$$

$$\Rightarrow \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} \underbrace{f(n)}_{ne^{-n}} \le \underbrace{f(1)}_{\frac{1}{e}} + \underbrace{\int_1^{\infty} f(x)\,dx}_{2e^{-1} = \frac{2}{e}}$$

$$\Rightarrow \boxed{\frac{2}{e} \le \sum_{n=1}^{\infty} ne^{-n} \le \frac{3}{e}}$$

Last time -

Finer version of the Comparison Test: called the Limit Comparison Test.

If $a_k \ge 0$ $\forall k$ ("for all k"), $b_k \ge 0$ $\forall k$

and $\lim_{k\to\infty} \frac{a_k}{b_k} = L \ne 0, \ne \infty$.

Then $\sum a_k$ converges $\Leftrightarrow$ $\sum b_k$ converges

$\sum a_k$ diverges $\Leftrightarrow$ $\sum b_k$ diverges.

Example Does $\displaystyle\sum_{n=1000}^{\infty} \frac{2n-3+n^5}{n^7+e^{-n}-3n^6}$ converge?

Limit comparison test:

Since $a_n = \dfrac{2n-3+n^5}{n^7+e^{-n}-3n^6} \geq 0$ for large n

and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p=2>1$)

And $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^2(a_n)$

$$= \lim_{n\to\infty} n^2\left(\frac{2n-3+n^5}{n^7+e^{-n}-3n^6}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2n^3-3n^2+n^7}{n^7+e^{-n}-3n^6}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\overset{\to 0}{\frac{2}{n^4}} - \overset{\to 0}{\frac{3}{n^5}} + 1}{1 + \underset{\to 0}{\frac{1}{e^n n^7}} - \underset{\to 0}{\frac{3}{n}}}\right) = 1$$

divide numerator & denom by n^7

(limit exists, not zero).

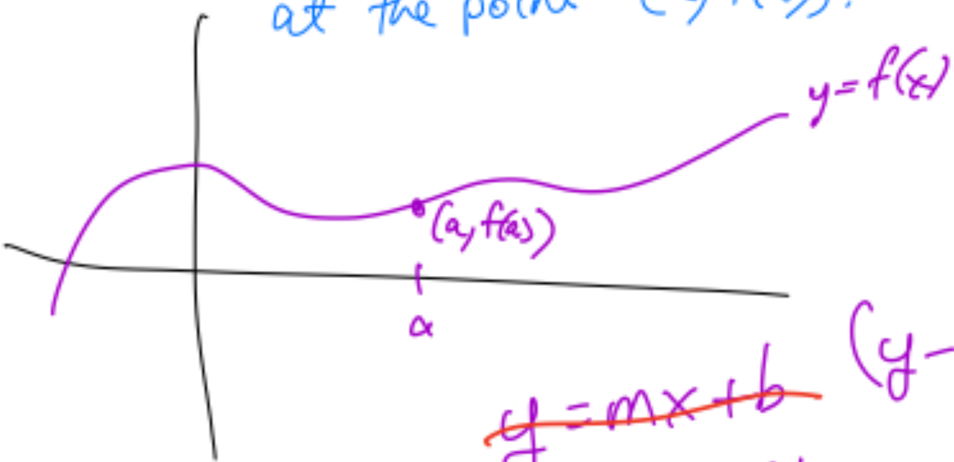
$\therefore$ Since $\displaystyle\sum_{n=1000}^{\infty} b_n$ converges, so

so does $\displaystyle\sum_{n=1000}^{\infty} a_n$. $\blacksquare$

Something Completely Different

Taylor Polynomials & Taylor series.

Idea: What is the equation of the tangent line of $y=f(x)$ at the point $(a, f(a))$?



$$\cancel{y=mx+b} \quad (y-y_0)=m(x-x_0)$$

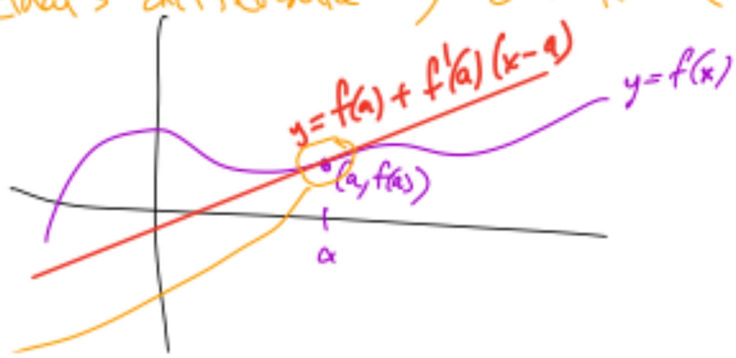
$y_0 = f(a)$, $x_0 = a$

$$m=f'(a).$$

$$\Rightarrow \quad y-f(a)=f'(a)(x-a)$$

$$\boxed{y=f(a)+f'(a)(x-a)}$$

formula for the tangent line of any function $f(x)$ (that's differentiable) at $x=a$.



for x really close to a

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}$$

1st degree Taylor polynomial of $f(x)$ @ $x=a$.

General formula for the n^{th} degree Taylor polynomial of $f(x)$ at $x=a$:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

n^{th} derivative of $f(x)$ at $x=a$.

Why does this work? Why is $T_n(x)$ very close to $f(x)$ near $x=a$.

$$T_n(a) = f(a) + f'(a)\underbrace{(a-a)}_{0} + \frac{f''(a)}{2!}\overbrace{(a-a)}^{0}{}^2 + \dots + \frac{f^{(n)}(a)}{n!}\underbrace{(a-a)}_{0}{}^n$$

$$= f(a). \checkmark$$

So values of $f(x)$ & $T_n(x)$ match at $x=a$.

$$T_n'(x) = f'(a) + \frac{f''(a)}{2!}2\cdot(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}n\cdot(x-a)^{n-1}$$

$$T_n'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

$$T_n''(x) = f''(a) + f'''(a)(x-a) + \frac{f''''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}.$$

plug in $x=a$.

$$\left.\begin{aligned} T_n'(a) &= f'(a) \\ T_n''(a) &= f''(a) \\ &\vdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}\right]$$

Taylor polynomial of degree n match all derivatives of $f(x)$ through the n^{th} derivative at $x=a$.

The Taylor series of $f(x)$ at $x=a$

$$T_\infty(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \cdots$$

$$\boxed{T_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

Taylor series of $f(x)$ at $x=a$.

$\boxed{\text{Example}}$. Find the 1st, 2nd, 3rd, etc Taylor polynomials of $y = \underset{f(x)}{\sin(x)}$ at $x=0$.

$$\begin{aligned}
& && x=0 \\
f(x) &= \sin(x) &&= 0 \\
f'(x) &= \cos(x) &&= 1 \\
f''(x) &= -\sin(x) &&= 0 \\
f'''(x) &= -\cos(x) &&= -1 \\
f^{iv}(x) &= \sin(x) &&= 0 \\
f^{(5)}(0) &= 1 \\
f^{(6)}(0) &= 0 \\
f^{(7)}(0) &= -1 \\
&\vdots
\end{aligned}$$

Taylor series of $\sin(x)$ at $x=0$ is

$$T_\infty(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$T_\infty(x) = 0 + 1x + 0 + \frac{-1}{3!}x^3 + 0 + 1\cdot\frac{x^5}{5!} + 0 + \frac{(-1)x^7}{7!}$$

$$\boxed{T_\infty(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$